# Toroidal Resonators and Waveguides of Arbitrary Cross Section

FERDINAND CAP

*Abstract*—After introducing a new method to solve Maxwell's equations using a complex electromagnetic field vector $F$, a rotational coordinate system $\xi, \theta, \varphi$ is introduced. In this coordinate system, the field vector components $F_\xi, F_\theta$ may be expressed by $F_\varphi$. This component can be obtained from a two-dimensional Helmholtz equation. Specifying $\xi, \theta$ by cylindrical coordinates $r, z$ the complex Maxwell equation curl $F=\gamma F$ is solved for the axisymmetric case $(\partial/\partial\varphi=0)$ and for the nonsymmetric case. The differential equations for magnetic field lines are solved and surfaces on which the normal component of $B$ and the tangential components of $E$ vanish are recognized as metallic walls of toroidal resonators of various arbitrary cross sections. In the Appendix the results of the new method are compared with well known results for circular cylindrical waveguides.

## I. Introduction

TOROIDAL RESONATORS having a high $Q$-value are of interest not only for microwave engineering, but also for light pipes and for the heating of toroidal plasmas by low-frequency electromagnetic waves. We have recently published solutions to Maxwell's equations for a torus with circular cross section [1], [2]. On the other hand it seems to be of interest to investigate also toroidal resonators and curved waveguides of arbitrary, e.g., elliptic or triangular cross section. Such cross sections are of interest, e.g., in plasma physics. In order to solve Maxwell's equations for a torus of arbitrary cross section we first present a new method to solve Maxwell's equations and the vector Helmholtz equation for arbitrary rotational coordinates.

The new method consists of three steps: 1) a complex electromagnetic field is defined in order to simplify Maxwell's equations; 2) for rotational coordinate systems describing three-dimensional toroidal configurations of arbitrary cross section, Maxwell's equations are solved by expressing two components of the complex electromagnetic fields by the third component; and 3) from the vector Helmholtz equation for the field first a fourth-order equation for the third field-component is derived which is then reduced to a second-order scalar Helmholtz equation. This equation is then solved for various practical examples.

## II. Complex Simplification of Maxwell's Equations

For a time-dependence $\sim\exp(i\omega t)$ Maxwell's equations may be written for vacuum

$$\operatorname{curl} \boldsymbol{E}=i\omega \boldsymbol{B} \tag{1}$$

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The author is with the Institute for Theoretical Physics, University of Innsbruck, Innsbruck, Austria.

$$\operatorname{curl} \boldsymbol{B}=\epsilon_0\mu_0 i\omega \boldsymbol{E} \tag{2}$$

$$\operatorname{div} \boldsymbol{E}=0 \tag{3}$$

$$\operatorname{div} \boldsymbol{B}=0. \tag{4}$$

For waveguides and empty resonators we have the boundary conditions

$$\boldsymbol{B}_n=0 \tag{5}$$

where $\boldsymbol{B}_n$ is the magnetic component normal to the metallic wall, and

$$\boldsymbol{E}_t=0 \tag{6}$$

where $\boldsymbol{E}_t$ is the electric component tangential to the highly conducting wall. Introducing a complex electromagnetic field $\boldsymbol{F}$ by

$$\boldsymbol{F}=\boldsymbol{E}-i\boldsymbol{B}/\sqrt{\epsilon_0\mu_0} \tag{7}$$

and adding (1)+(2) as well as (3)+(4) we may write Maxwell's equation in the compact complex form

$$\operatorname{curl} \boldsymbol{F}=\omega\sqrt{\epsilon_0\mu_0}\,\boldsymbol{F} \tag{8}$$

$$\operatorname{div} \boldsymbol{F}=0. \tag{9}$$

(By the way, if $\boldsymbol{F}$ is replaced by $\boldsymbol{B}$, these equations are identical with the equations describing force-free plasma containment [4]).

A reviewer who has been so kind to evaluate this paper has drawn our attention to the fact that the use of a complex field vector were not new. It had been discussed in Stratton, *Electromagnetic Theory*, a book not available here in Innsbruck. A second reviewer has been so kind to suggest the use of bicomplex variables. It is the feeling of the author that this could yield a more flexible and elegant way but would give the same results. Anyway, both reviewers are thanked for their remarks.

Due to (9) we may make the ansatz

$$\boldsymbol{F}=\operatorname{curl} \boldsymbol{P}+\omega\sqrt{\epsilon_0\mu_0}\,\boldsymbol{P} \tag{10}$$

$$\operatorname{div} \boldsymbol{P}=0 \tag{11}$$

where the complex vector $\boldsymbol{P}$ is nothing else than $-i\boldsymbol{A}/\sqrt{\epsilon_0\mu_0}$, where $\boldsymbol{A}$ is the usual magnetic vector potential. It is easy to show that $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{F}$, and $\boldsymbol{P}$ satisfy a vector Helmholtz equation

$$\nabla^2\boldsymbol{F}+\omega^2\epsilon_0\mu_0\boldsymbol{F}=0 \tag{12}$$

$$\nabla^2\boldsymbol{P}+\omega^2\epsilon_0\mu_0\boldsymbol{P}=0. \tag{13}$$

This is, however, not of interest for the following.

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## III. Rotational Coordinate System

We now introduce three-dimensional curvilinear orthogonal rotational coordinates $\xi, \theta, \varphi$ related to Cartesian coordinates by

$$x = g(\xi,\theta)\cos\varphi$$
$$y = g(\xi,\theta)\sin\varphi$$
$$z = l(\xi,\theta). \tag{14}$$

See Fig. 1.

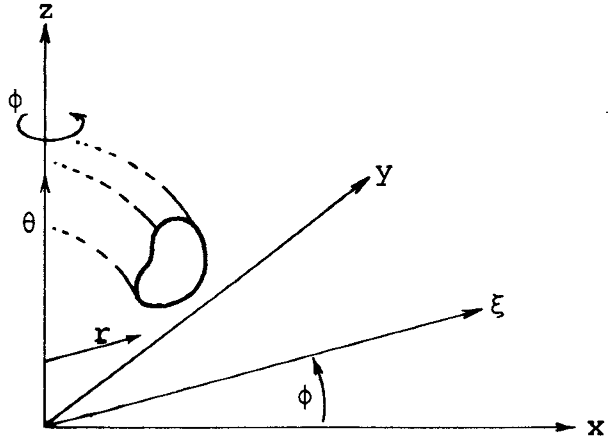

Fig. 1. Rotational coordinates.

The functions $g(\xi,\theta)$, $l(\xi,\theta)$ will be specified later. They describe the arbitrary cross section of the torus. The system $\xi, \theta, \varphi$ is obtained by rotation around the $z$-axis of the two-dimensional system $\xi, \theta$. $\varphi$ is the rotation angle. The scale factors are given by

$$h_\xi^2 = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 = \left(\frac{\partial g}{\partial \xi}\right)^2 + \left(\frac{\partial l}{\partial \xi}\right)^2 \tag{15}$$

$$h_\theta^2 = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial g}{\partial \theta}\right)^2 + \left(\frac{\partial l}{\partial \theta}\right)^2 \tag{16}$$

$$h_\varphi^2 = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = g^2. \tag{17}$$

According to Zagrodzinski [3] we now assume that the toroidal cross section can be generated by conformal mapping of the $x, z$-plane on the $\xi, \theta$-plane or that $\xi + i\theta = f(g + il) = f(x+iz)$, such that $g$ and $l$ satisfy the Cauchy–Riemann equations

$$\frac{\partial g}{\partial \xi} = \frac{\partial l}{\partial \theta} \qquad \frac{\partial g}{\partial \theta} = -\frac{\partial l}{\partial \xi}. \tag{18}$$

We then have from (15), (16)

$$h_\xi^2 = h_\theta^2 = \left(\frac{\partial g}{\partial \xi}\right)^2 + \left(\frac{\partial g}{\partial \theta}\right)^2 = h^2. \tag{19}$$

Furthermore we obtain from (18), (19) the results

$$(\operatorname{grad} l)^2 = \left(\frac{1}{h_\xi}\frac{\partial l}{\partial \xi}\right)^2 + \left(\frac{1}{h_\theta}\frac{\partial l}{\partial \theta}\right)^2 + \left(\frac{1}{h_\varphi}\frac{\partial l}{\partial \varphi}\right)^2 = 1 \tag{20}$$

$$(\operatorname{grad} h_\varphi)^2 = (\operatorname{grad} g)^2 = 1 \tag{21}$$

and

$$\nabla^2 h_\varphi = \nabla^2 g = g^{-1}. \tag{22}$$

Since Maxwell's equations are linear and since their solution must be periodic in $\varphi$, we assume for all components of $F$ a $\varphi$-dependence $\sim \Sigma_m A_m \exp(im\varphi)$. By combining linearly the three components of (8) we obtain in analogy to [3] the following equations for the first two components:

$$F_\xi = \frac{1}{Mh}\left(+im\frac{\partial F_\varphi g}{\partial \xi} - g\omega\sqrt{\epsilon_0\mu_0}\,\frac{\partial F_\varphi g}{\partial \theta}\right) \tag{23}$$

and

$$F_\theta = \frac{1}{Mh}\left(im\frac{\partial F_\varphi g}{\partial \theta} + g\omega\sqrt{\epsilon_0\mu_0}\,\frac{\partial F_\varphi g}{\partial \xi}\right) \tag{24}$$

where $M \equiv g^2\omega^2\epsilon_0\mu_0 - m^2$ and where $g$, $h$ are given by the form of the arbitrary toroidal cross section. We are thus able to calculate $F_\xi$ and $F_\theta$, if $F_\varphi$ is known. If $F_\xi$ and $F_\theta$ are inserted into the third component equation of (8) or into (9), a very complicated equation of second order containing only $F_\varphi$ can be derived (see, e.g., (40)). This equation is *not* of the form of a scalar Helmholtz equation. There is, however, a more simple way to find $F_\varphi$.

## IV. Helmholtz Equation for $F_\varphi$.

According to [3] we consider the azimuthal component $F_\varphi$ and decompose it into Cartesian components

$$F_\varphi = -F_x\sin\varphi + F_y\cos\varphi. \tag{25}$$

Application of $\nabla^2$ to (25) yields

$$\nabla^2 F_\varphi = -\left(\omega^2\epsilon_0\mu_0 + g^{-2}\right)F_\varphi - 2g^{-2}\left(\frac{\partial B_x}{\partial \varphi}\cos\varphi + \frac{\partial B_y}{\partial \varphi}\sin\varphi\right). \tag{26}$$

Differentiating (25) with respect to $\varphi$ we obtain

$$\frac{\partial F_\varphi}{\partial \varphi} = -\frac{\partial F_x}{\partial \varphi}\sin\varphi + \frac{\partial F_y}{\partial \varphi}\cos\varphi - F_r \tag{27}$$

where

$$F_r = F_x\cos\varphi + F_y\sin\varphi \tag{28}$$

is the radial component in cylinder coordinates $r, z, \varphi$. On the other hand

$$\frac{\partial F_r}{\partial \varphi} = \frac{\partial F_x}{\partial \varphi}\cos\varphi + \frac{\partial F_y}{\partial y}\sin\varphi + F_\varphi \tag{29}$$

so that (26) may be written

$$\tfrac{1}{2}g^2\left(\nabla^2 F + \omega^2\epsilon_0\mu_0 F_\varphi - g^{-2}F_\varphi\right) = -im\,F_r. \tag{30}$$

Similarly application of $\nabla^2$ on (28) yields

$$\tfrac{1}{2}g^2\left(\nabla^2 F_r + \omega^2\epsilon_0\mu_0 F_r - g^{-2}F_r\right) = im\,F_\varphi. \tag{31}$$

Elimination of $F_r$ and use of $\nabla^2=\nabla_2^2-g^{-2}m^2$ yields an equation of fourth order for $F_\varphi$

$$\left[\left(\nabla_2^2-g^{-2}m^2+\omega^2\epsilon_0\mu_0-g^{-2}\right)\cdot\left(\nabla_2^2-g^{-2}m^2+\omega^2\epsilon_0\mu_0-g^{-2}\right)-4g^{-4}m^2\right]F_\varphi=0 \tag{32}$$

where

$$\nabla_2^2=\frac{1}{h^2g}\left[\frac{\partial}{\partial\xi}\left(g\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\theta}\left(g\frac{\partial}{\partial\theta}\right)\right]. \tag{33}$$

Thus, according to (32) $F_\varphi=C_1F_\varphi^++C_2F_\varphi^-$ is determined by the two scalar Helmholtz equations

$$\nabla_2^2F_\varphi^++\omega^2\epsilon_0\mu_0F_\varphi^+-g^{-2}(m+1)^2F_\varphi^+=0 \tag{34}$$

$$\nabla_2^2F_\varphi^-+\omega^2\epsilon_0\mu_0F_\varphi^--g^{-2}(m-1)^2F_\varphi^-=0 \tag{35}$$

and by the boundary conditions (5), (6).

The constants $C_1, C_2$ must be determined in such a way that $F_\varphi$ given by (34), (35) and $F_\xi, F_\theta$ given by (23), (24) satisfy (8), (9).

## V. Cylindrical Coordinates

To solve the equations (34)–(35) we now must specify the coordinate system. If we chose quasitoroidal coordinates $\rho,\theta,\varphi$ by $g=R\eta$, $h=R\rho$, $l=R\rho\sin\theta$, $\eta=1-\rho\cos\theta$ we obtain a torus with exactly circular cross section and we come back to the problem investigated in [1], [2]. Since equation (34) is not separable in these coordinates even for $m=0$ [4] we again have to use series approximations. We therefore choose now a coordinate system in which (34) is separable. In cylinder coordinates $r,z,\varphi$ we have $\xi=r$, $\theta=z$, $\varphi=\varphi$ and $g=r$, $l=z$, $h=1$. For the $z$-dependence we choose $\exp(ikz)$. Then (34), (35) read

$$\frac{d^2F_\varphi^\pm}{dr^2}+\frac{1}{r}\frac{dF_\varphi^\pm}{dr}+(\gamma^2-k^2)F_\varphi^\pm-\frac{(m\pm1)^2}{r^2}F_\varphi^\pm=0 \tag{36}$$

where $\gamma^2=\omega^2\epsilon_0\mu_0$. The solution $F_\varphi=C_1F_\varphi^++C_2F_\varphi^-$ is given by

$$F_\varphi(r)=\left[C_1Z_{m+1}\left(\sqrt{\gamma^2-k^2}\,r\right)+C_2Z_{m-1}\left(\sqrt{\gamma^2-k^2}\,r\right)\right]. \tag{37}$$

Here the $Z_p$ are cylindrical functions. Sums over $m$ and $k$ are omitted. From (23) and (24) we obtain

$$F_r=\frac{1}{r^2\gamma^2-m^2}\left[im\frac{\partial rF_\varphi}{\partial r}-r\gamma\frac{\partial rF_\varphi}{\partial z}\right] \tag{38}$$

$$F_z=\frac{1}{r^2\gamma^2-m^2}\left[im\frac{\partial rF_\varphi}{\partial z}+r\gamma\frac{\partial rF_\varphi}{\partial r}\right]. \tag{39}$$

Inserting (37)–(39) into (8) we find that the $r$ and $z$-components are identically satisfied, but the $\varphi$-component yields for a $z$-dependence $\exp(ikz)$ ($'=d/dr$)

$$F_\varphi''+\frac{1}{r}F_\varphi'+F_\varphi(\gamma^2-k^2)+\frac{1}{r^2}F_\varphi(1-m^2)-\frac{2m^2}{\gamma^2r^2-m^2}\frac{F_\varphi'}{r}+\frac{2\gamma mk}{\gamma^2r^2-m^2}F_\varphi-\frac{2\gamma^2}{\gamma^2r^2-m^2}F_\varphi=0. \tag{40}$$

According to Section IV, this equation must have the same solution as (8) and (36) if the constants $C_1$ and $C_2$ are chosen appropriately. To see this we solve (8) in cylindrical coordinates:

$$F_\varphi=A_{mk}e^{im\varphi+ikz}\left[-\frac{mk}{r}Z_m\left(\sqrt{\gamma^2-k^2}\,r\right)-\gamma Z_m'\left(\sqrt{\gamma^2-k^2}\,r\right)\right] \tag{41}$$

$$F_r=A_{mk}ie^{im\varphi+ikz}\left[+\frac{m\gamma}{r}Z_m\left(\sqrt{\gamma^2-k^2}\,r\right)+kZ_m'\left(\sqrt{\gamma^2-k^2}\,r\right)\right] \tag{42}$$

$$F_z=A_{mk}e^{im\varphi+ikz}\left[(\gamma^2-k^2)Z_m\left(\sqrt{\gamma^2-k^2}\,r\right)\right]. \tag{43}$$

Identifying (41) with (37) and using

$$Z_m'\left(\sqrt{\gamma^2-k^2}\,r\right)=\sqrt{\gamma^2-k^2}\,Z_{m-1}-\frac{m}{r}Z_m \tag{44}$$

as well as

$$\frac{m}{r}Z_m\left(\sqrt{\gamma^2-k^2}\,r\right)=\frac{\sqrt{\gamma^2-k^2}}{2}\cdot\left[Z_{m+1}\left(\sqrt{\gamma^2-k^2}\,r\right)+Z_{m-1}\left(\sqrt{\gamma^2-k^2}\,r\right)\right] \tag{45}$$

we obtain

$$C_1=\frac{1}{2}A_{mk}\sqrt{\gamma^2-k^2}\,(\gamma-k)$$

$$C_2=\frac{1}{2}A_{mk}\sqrt{\gamma^2-k^2}\,(-\gamma-k). \tag{46}$$

It is remarkable that the same expression ((37)≡(41)) solves two quite different equations, namely (36) and (40). When we insert (37) or (41) into (38) and (39) we obtain (42) and (43) as it must be.

## VI. The Axisymmetric Mode

For $\varphi$-independence, i.e., for $m=0$ we have the axisymmetric mode. In this case (23) and (24) become

$$F_\xi^0=-\frac{1}{hg\gamma}\frac{\partial F_\varphi g}{\partial\theta} \tag{47}$$

$$F_\theta^0=\frac{1}{hg\gamma}\frac{\partial F_\varphi g}{\partial\xi} \tag{48}$$

and from (41) and (37) we have in cylindrical coordinates

$$F_\varphi^0=-A_{0k}\gamma\cos kz Z_0'\left(\sqrt{\gamma^2-k^2}\,r\right)=A_{0k}\sqrt{\gamma^2-k^2}\,\gamma Z_1\left(\sqrt{\gamma^2-k^2}\,r\right)\cos kz \tag{49}$$

where (44), (46) have been used.

Since we want to describe a toroidal configuration, the boundary is not given by $r=r_0$ (which would correspond to a circular cylindrical waveguide in the $z$-direction) but by a

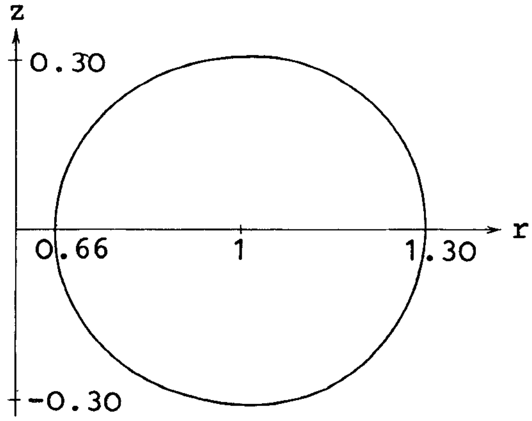


Fig. 2. Nearly circular cross section.

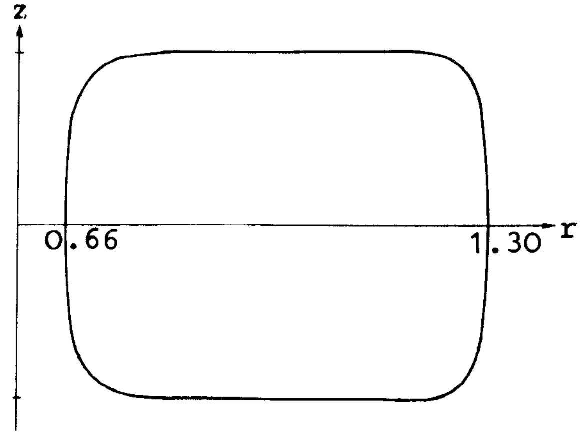


Fig. 4. Nearly rectangular cross section.

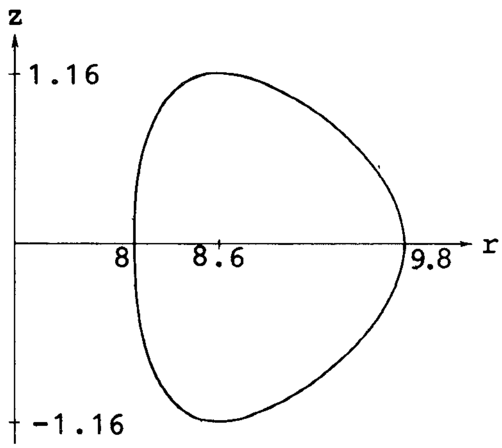


Fig. 3. D-shaped cross section.

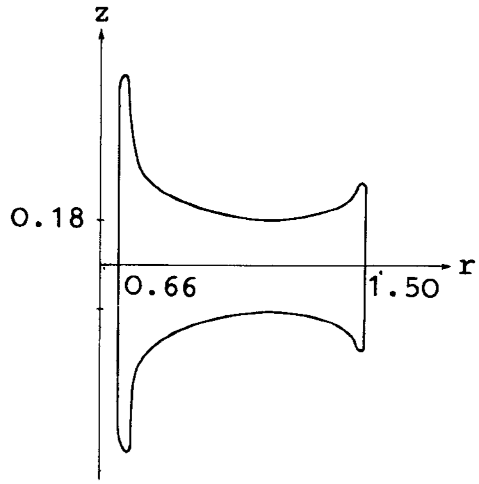


Fig. 5. Hollow cross section.

TABLE I
MEASURES, IN CENTIMETERS, AND EIGENFREQUENCIES OF EMPTY TOROIDAL RESONATORS WITH CROSS SECTIONS SHOWN IN FIGS. 2–7

| Fig | $k_1$ | $r_1$ | $z_1$ | $r_2$ | $z_2$ | $r_3$ | $z_3$ | $r_4$ | $z_4$ | $r_5$ | $z_5$ | f in GHz | D | $b_o$ | $c_o$ | $b_1$ | $\gamma$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0.66 | 0 | 1.30 | 0 | 1 | 0.30 | 0.80 | 0.244 | 1.20 | 0.224 | 6.8655 | -1.1301 | 0.08675 | -0.00977 | -0.26270 | 1.4393 |
| 3 | 1.5 | 8 | 0 | 9.8 | 0 | 8.6 | 1.16 | 8.3 | 1.07 | 9 | 1.02 | 8.5721 | 3.8741 | 0.7185 | 0.4405 | 1.3299 | 1.7971 |
| 4 | 1 | 0.66 | 0 | 1.30 | 0 | 1 | 0.26 | 0.82 | 0.26 | 1.20 | 0.26 | 22.1389 | 0.3915 | 0.6705 | -0.8457 | -0.6288 | +4.6412 |
| 5 | 1 | 0.66 | 0 | 1.50 | 0 | 1 | 0.18 | 0.80 | 0.26 | 1.20 | 0.16 | 19.6275 | 0.3404 | 0.6483 | -0.8469 | -0.5614 | 4.1147 |
| 6 | 1.62 | 0.95 | 0 | integration, $\Delta\varphi = \pi/20$, $\varphi_o = 2\pi$ | | | | | | | | $b_2 = -0.01$ | | -0.2506 | 0.1805 | 0.5076 | 2.04925 |
| 7 | 1.62 | 0.95 | -0,01 | integration, $\Delta\varphi = \pi/20$, $\varphi_o = 3\pi/2$ | | | | | | | | $c_2 = -0.015$, $c_1 = 1$ | | -0.2506 | 0.1805 | 0.5076 | 2.04925 |

function

$$f(r,z)=\text{const.} \tag{50}$$

This curve in the $r$, $z$-plane (which rotates through $\varphi$ around the $z$-axis) must be closed. On its circumference the boundary conditions must be satisfied. If we want to describe a torus of major radius $R$ with, e.g., circular cross section of minor radius $\rho_0$ we have

$$f(r,z)\equiv(r-R)^2+z^2-\rho_0^2=0. \tag{51}$$

This form is however not useful since (5) and (6) have to be satisfied along this curve which does not coincide with a coordinate surface. It is therefore expedient to describe $f(r,z)$ by the same functions by which the components of $\boldsymbol{F}$ are expressed.

According to (5) the $\boldsymbol{B}$-field lines are tangential to the wall of the resonator. We take therefore the imaginary parts of the constants $A_{0k}$ in (49) and using cylindrical coordinates agains we write (49), (42) or (47), and (43) or

(48) for the first two modes ($m=0, k=0, k_1$):

$$B_\varphi^0(r,z)=\gamma^2[b_0J_1(\gamma r)+c_0Y_1(\gamma r)]+\gamma\sqrt{\gamma^2-k_1^2}\cdot\left[b_1J_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)+c_1Y_1\left(\sqrt{\gamma^2-k^2}\,r\right)\right]\cos k_1z \quad (52)$$

$$B_r^0(r,z)=k_1\sqrt{\gamma^2-k_1^2}\cdot\left[b_1J_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)+c_1Y_1\left(\sqrt{\gamma^2-k^2}\,r\right)\right]\sin k_1z \quad (53)$$

$$B_z^0(r,z)=\gamma^2[b_0J_0(\gamma r)+c_0Y_0(\gamma r)]+\left(\gamma^2-k_1^2\right)\cdot\left[b_1J_0\left(\sqrt{\gamma^2-k^2}\,r\right)+c_1Y_0\left(\sqrt{\gamma^2-k^2}\,r\right)\right]\cos k_1z \quad (54)$$

where $J_p$ and $Y_p$ are Bessel and Neumann functions, respectively.

Now the differential equations for the field lines in the $r, z$-plane are

$$\frac{dr}{B_r^0}=\frac{dz}{B_z^0}. \quad (55)$$

Inserting for $B_r^0$ and $B_z^0$ from (53), (54) we may integrate (55). There is however another possibility. We can insert the imaginary parts of (47) and (48) into (55). This gives

$$\frac{\partial\left(B_\varphi^0 r\right)}{\partial r}dr+\frac{\partial\left(B_\varphi^0 r\right)}{\partial z}dz=d\left(B_\varphi^0 r\right)=0. \quad (56)$$

Thus the lines $B_\varphi^0 r=\text{const}=D$ are *identical* in form with the $B_r^0, B_z^0$ field lines in the $r, z$-plane, i.e., identical with the cross section of the toroidal resonator. In order to obtain a toroidal resonator of major radius $R$ and nearly circular cross section of minor radius $\rho_0$ of the form (50) or (51) we have to determine the constants $b_0, c_0, b_1, c_1, D$ in $B_\varphi^0$ according to (52) from

$$z=\frac{1}{k_1}\arccos\cdot\left[\frac{D-b_0r\gamma^2J_1(\gamma r)-c_0\gamma^2rY_1(\gamma r)}{\gamma\sqrt{\gamma^2-k_1^2}\left(b_1rJ_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)+c_1rY_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)\right)}\right]. \quad (57)$$

By choosing for a given $k_1$ a set of coordinates $z_i=z(r_i)$, $i=1,\cdots,5$ we may generate various arbitrary cross sections. For five given values $r_i$, $z_i$ equation (57) yields five homogeneous linear equations for the five unknown constants $D, b_0, c_0, b_1, c_1$. In order that this system possess a nontrivial solution, its fifth-order determinant must vanish. The transcendental equation obtained yields the eigenvalue $\gamma$. The various zeroes $\gamma_n$ represent the radial mode number $n$. As in the usual theory (see Appendix) the eigenvalue $\gamma$ and thus the eigenfrequency $f=\gamma/2\pi\sqrt{\epsilon_0\mu_0}$ is determined by the measures $(r_i, z_i)$ of the resonator. The equation has been solved using a Hewlett-Packard table top computer 9825A. The results are summarized in Table I and the cross sections obtained are depicted in Figs. 2–5. In the axisymmetric case the boundary conditions are automatically satisfied since $\boldsymbol{B}$ is tangential to the torus wall. Other cross sections can also be found by introducing appropriate coordinate systems $\xi, \theta$ and integrating (34), (35). Since these equations are however not separable, a numerical integration is necessary. This seems, however, not to be necessary, since more modes than the two modes in (57) allow to determine more parameters and to fix more $z_i=z(r_i)(i>5)$ so that nearly arbitrary cross sections may be produced with more modes. If more modes are taken into account then (5) is better satisfied. (In our case $E_\varphi$ is only part of $E_t$.)

## VI. The Nonaxisymmetric Case

For $m\neq 0$, the procedure described in the last chapter is no longer possible. Since the curvature of the torus abolishes the degeneracy of the cylindrical modes [5], [6] we now have the following modes:

$$F_{mkn}^s, F_{mkn}^a, \qquad \text{where}\begin{cases} m=0,1,2,\cdots,(\varphi)\\ n=1,2,3,\cdots,(r)\\ k=0,1,2,\cdots,(z)\end{cases}$$

where the superscript $s(a)$ indicates symmetry or antisymmetry under reflection with respect to the equatorial plane of the torus. From (41) to (43) a simple three-dimensional solution is obtained ($m=0,1$, $k=0, k_1$, $n=1$)

$$\begin{aligned}B_\varphi(r,z,\varphi)=&\gamma^2[b_0J_1(\gamma r)+c_0Y_1(\gamma r)]\\ &(m=0,k=0)\\ &+\gamma\sqrt{\gamma^2-k_1^2}\left[b_1J_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)+c_1Y_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)\right]\cos k_1z\\ &(m=0,k=k_1)\\ &+\gamma\left[-\gamma b_2J_0(\gamma r)-\gamma c_2Y_0(\gamma r)+\frac{b_2}{r}J_1(\gamma r)+\frac{c_2}{r}Y_1(\gamma r)\right]\cos\varphi\\ &(m=1,k=0)\end{aligned} \quad (58)$$

$$\begin{aligned}B_r(r,z,\varphi)=&k_1\sqrt{\gamma^2-k_1^2}\left[b_1J_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)+c_1Y_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)\right]\sin k_1z\\ &(m=0,k=k_1)\\ &-\left\{\frac{\gamma}{r}[b_2J_1(\gamma r)+c_2Y_1(\gamma r)]\sin\varphi\right\}\\ &(m=1,k=0)\end{aligned} \quad (59)$$

$$\begin{aligned}B_z(r,z,\varphi)=&\gamma^2[b_0J_1(\gamma r)+c_0Y_1(\gamma r)]\\ &(m=0,k=0)\\ &+\left(\gamma^2-k_1^2\right)\left[b_1J_0\left(\sqrt{\gamma^2-k_1^2}\,r\right)+c_1Y_1\left(\sqrt{\gamma^2-k_1^2}\,r\right)\right]\cos k_1z\\ &(m=0,k=k_1)\\ &+\gamma^2[b_2J_1(\gamma r)+c_2Y_1(\gamma r)]\cos\varphi\\ &(m=1,k=0).\end{aligned} \quad (60)$$

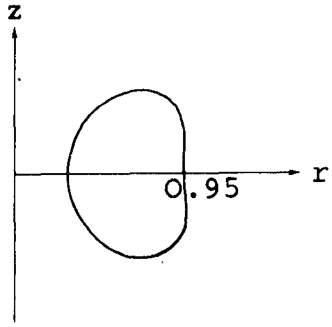


Fig. 6. Toroidal waveguide, meridional cut at $\varphi_0=2\pi$.

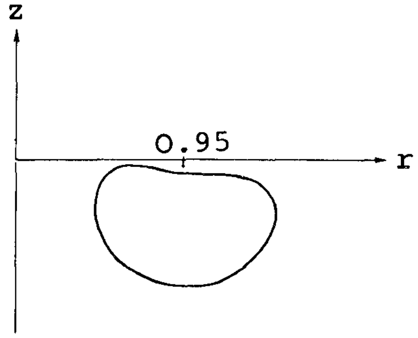


Fig. 7. Toroidal waveguide, meridional cut at $\varphi_0=3\pi/2$.

Now we have to integrate

$$\frac{dr}{B_r}=\frac{dz}{B_z}=\frac{rd\varphi}{B_\varphi} \tag{61}$$

in order to find the field lines and the form of the wall of the toroidal waveguide. We write (61) in the form

$$\frac{dz}{d\varphi}=\frac{rB_z}{B_\varphi} \qquad \frac{dr}{d\varphi}=\frac{rB_r}{B_\varphi} \tag{62}$$

in order to obtain $z(\varphi), r(\varphi)$. Integrating (62) for $0\leqslant\varphi\leqslant 2\pi, 2\pi\leqslant\varphi\leqslant 4\pi, 4\pi\leqslant\varphi\leqslant 6\pi$, etc., we may search how for a fixed $\varphi_0=\text{const}$ (meridional cut) the passing points of a given field line (defined by its initial conditions $r_i, z_i$) can be found. There are two types of field lines: a) field lines meeting their first passing point after one or several revolutions—we then have periodicity in $\varphi$ after $n\cdot 2\pi$ (i.e., after $n$ revolutions); b) field lines never meeting their first passing point through the meridional cut—these are just the field lines in which we are interested because they form asymptotically closed surfaces on which $B_n=0$, so that these surfaces can be identified with metallic surfaces forming the wall of a toroidal waveguide. The cuts at $\varphi_0=0, \varphi_0=\pi/4$, etc., exhibit the cross sections of the torus. In order to follow a field line going around and around, it is necessary to make an integration over a large interval $0\leqslant\varphi\leqslant n2\pi$. For example, $n=50$, we obtain 51 crossing points of the field line in the meridional cut, e.g., at $\varphi_0=0$. Repeating the same procedure for $\varphi_0=\pi/4, \pi/2, 3\pi/2$, etc., we obtain a series of meridional cuts. Putting together these cuts we get a picture of the spatially helically wound toroidal waveguide. (Such waveguides and resonators are used in plasma physics, e.g., high beta stellarators.) As in the axisymmetric case the shape of the torus cross section (i.e., the $r_i, z_i$ and the constants $b_i, c_i$) determines $\gamma$ and thus the eigenfrequency. For small $b_2, c_2$ the eigenfrequency is nearly the same as in the axisymmetric case. Computer facilities available here did not allow the calculation of the distribution of the frequency due to $b_2\neq c_2\neq 0$. Facilities here did not permit integrations over such long intervals, but a 30-h run on a desk top Hewlett Packard 9825A resulted in Figs. 6 and 7 on which the helical rotation and slight modification of the cross section can be seen. Toruses of cross sections like this one shown in Fig. 2 can be obtained for $m\neq 0$ if one assumes that $b_2$ and $c_2$ are very small, i.e., $<0.01$.

## Appendix

Since the method to work with complex electromagnetic fields (based partially on an earlier work of the author [4]) is unusual we will apply the method to a very well known example. We consider a circular cylindrical waveguide [8] in which the wave propagates into the $z$-direction (and not in the $\varphi$-direction as in this paper). The solution of the complex Maxwell equations (10), (11) is then given again by (41)–(43). Decomposing the complex field $\boldsymbol{F}$ according to (7) we obtain ($\gamma=\omega\sqrt{\epsilon_0\mu_0}$)

$$E_\varphi=-\gamma J_m'\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right) \tag{63}$$

$$B_\varphi=-\frac{im\,k\gamma}{\omega r}J_m\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right)\sim E_r \tag{64}$$

$$E_r=\frac{im\gamma}{r}J_m\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right) \tag{65}$$

$$B_r=-\frac{k}{\omega}J_m'\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right)\sim E_\varphi \tag{66}$$

$$E_z=0 \tag{67}$$

$$B_z=i\frac{\gamma}{\omega}\left(\gamma^2-k^2\right)J_m\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right). \tag{68}$$

This solution satisfies Maxwell's equations and corresponds to the TE wave. (If Maxwell's equations are solved directly, one has $E_\varphi=-iJ_m'$ instead of $-\gamma J_m'$, etc.) (63)–(68) satisfy at $r=r_0$ the boundary condition the TE wave. At $r=r_0$, it satisfies the boundary condition $E_z=0$

$$B_r(r_0)=0=E_\varphi(r_0)\ \text{or}\ J_m'\left(\sqrt{\gamma^2-k^2}\,r\right)=0 \tag{69}$$

which determines $\kappa$. The other solution is

$$E_\varphi=-\frac{mk}{r}J_m\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right) \tag{70}$$

$$B_\varphi=-\frac{i\gamma^2}{\omega}J_m'\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right)\sim E_r \tag{71}$$

$$E_r=ikJ_m'\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right) \tag{72}$$

$$B_r=-\frac{m\gamma^2}{\omega r}J_m\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right)\sim E_\varphi \tag{73}$$

$$E_z=\left(\gamma^2-k^2\right)J_m\left(\sqrt{\gamma^2-k^2}\,r\right)\exp\left(im\varphi+ikz\right) \tag{74}$$

$$B_z=0 \tag{75}$$

and corresponds to the TM wave. At $r=r_0$, it satisfies the boundary condition $E_z = 0$

$$B_r(r_0) = 0 = E_\varphi(r_0) \text{ or } J_m\left(\sqrt{\gamma^2 - k^2}\, r_0\right) = 0. \quad (76)$$

We consider now the case $m = 0$. Comparing (64), (66), (68), (71), (73), and (75) with (52)–(54) we find that the latter solution corresponds to a superposition of the TE parts of $B_r$ and $B_z$ and the TM part of $B_\varphi$. We use, however, only the last term giving from $rB_\varphi = \text{const}$ for the "mode" $k = 0$

$$rJ_1(\gamma r) = \text{const} \quad (77)$$

(corresponding to a waveguide with cross section $r = \text{const}$) and for $k = 1$ ($TM_{01}$ mode) we obtain

$$rJ_1\left(\sqrt{\gamma^2 - 1}\, r\right) \cos z = \text{const} \quad (78)$$

which describes the surfaces on which $B_r = E_\varphi = 0$ and into which metallic walls may be inserted without disturbing the field patterns.

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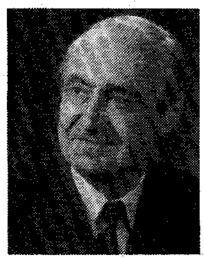

**Ferdinand Cap** was born in 1924 near Vienna, Austria. He studied physics, mathematics, and chemistry from 1942–1946 at the University of Vienna, Vienna, Austria.

He became a Lecturer at the University of Innsbruck, Innsbruck, Austria, in 1949 after a stay at the Zürich Technical University. He was promoted to Assistant Professor in 1955, Associate Professor with tenure in 1957, and Full Professor in 1960. In 1967 he was Guest Professor for plasma physics in New York. In 1971 he became a Senior Research Associate of NASA at Goddard Space Center and in 1979 he was a visitor at the Princeton Plasma Physics Laboratory. For 10 years he was Austria's Scientific Representative at the UN in New York. He has published more than 100 papers in physics and several textbooks about nuclear reactors and plasma physics.

# Analysis and Design of $TE_{11}$-to-$HE_{11}$ Corrugated Cylindrical Waveguide Mode Converters

GRAEME L. JAMES

***Abstract*— A theoretical parametric study is given of a $TE_{11}$-to-$HE_{11}$ mode converter consisting of a section of cylindrical corrugated waveguide with varying slot depth. The analysis makes use of modal field-matching techniques to determine the scatter matrix of the mode converter from which we deduce its propagation properties. It is shown that a mode converter consisting of only five slots achieves a return loss better than 30 dB over the band $2.7 < ka < 3.8$ (where $a$ is the internal radius of the waveguide) with the $HE_{11}$ mode in the balanced condition at $ka = 2.9$. The predicted results are in very good agreement with experimental data.**

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The author is with the Division of Radiophysics, CSIRO, P.O. Box 76, Epping, N.S.W. 2121, Australia.

## I. Introduction

In designing corrugated horns which use a section of cylindrical corrugated waveguide at the input, it is necessary to study the transition from a smooth-walled cylindrical waveguide supporting the $TE_{11}$ mode to a corrugated cylindrical waveguide where the $HE_{11}$ hybrid mode is supported. With the corrugated surface represented by

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